

ON THE EXISTENCE OF SIGN CHANGING BOUND STATE SOLUTIONS OF A QUASILINEAR EQUATION

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ABSTRACT. In this paper we establish the existence of bound state solutions having a prescribed number of sign change for

$$(P) \quad \Delta_m u + f(u) = 0, \quad x \in \mathbb{R}^N, N \geq m > 1,$$

where $\Delta_m u = \nabla \cdot (|\nabla u|^{m-2} \nabla u)$. Our result is new even for the case of the Laplacian ($m = 2$).

1. INTRODUCTION AND MAIN RESULTS

In this paper we establish the existence of higher bound state solutions to

$$(P) \quad \Delta_m u + f(u) = 0, \quad x \in \mathbb{R}^N, N \geq m > 1,$$

where $\Delta_m u = \nabla \cdot (|\nabla u|^{m-2} \nabla u)$. To this end we consider the radial version of (P), that is

$$\begin{aligned} (\phi_m(u'))' + \frac{N-1}{r} \phi_m(u') + f(u) &= 0, \quad r > 0, \quad N \geq m > 1, \\ u'(0) &= 0, \quad \lim_{r \rightarrow \infty} u(r) = 0, \end{aligned} \tag{1.1}$$

where $\phi_m(x) = |x|^{m-2}x$, $x \neq 0$, and $\phi_m(0) = 0$.

Any nonconstant solution to (1.1) is called a bound state solution. Bound state solutions such that $u(r) > 0$ for all $r > 0$, are referred to as a first bound state solution, or a ground state solution.

The existence of a first bound state for (P) has been established by many authors under different regularity and growth assumptions on the nonlinearity f , both for the Laplacian operator and the degenerate Laplacian operator, see for example [AP1, AP2], [BL1] and [FR] in the case of a regular f ($f \in C[0, \infty)$) for the case of the semilinear equation, and [FLS], [GST] and [FG] for both the singular and regular case in the quasilinear situation. In the case of the Laplacian, Berestycki and Lions in [BL2], proved the existence of infinitely many radially symmetric bound state solutions by using variational methods when f is an odd function satisfying very mild assumptions, but the existence of solutions with prescribed number of zeros was as an open question for many years. Jones and Küpper in [JK] gave a positive answer to this question using a dynamical systems approach and the Conley index for an f which at infinity grows like $|u|^\sigma u$, with $\sigma + 1 < (N + 2)/(N - 2)$ and f is not

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necessarily odd. In [McLTW], for the Laplacian operator and f satisfying a similar superlinear type condition at infinity, the authors prove the existence of bound states of any order by means of a shooting method and a scaling argument. Grillakis, in [G], studied the same problem for the Laplacian operator using the degree of a map from a Banach space to itself, allowing less smoothness in the nonlinear term, imposing a different subcritical restriction at infinity, and having a more restrictive growth condition for f at 0, namely, that f is Lipschitz at 0. Finally we mention the work in [BDO], where the authors treat the same problem for the Laplacian operator with the specific nonlinearity $u - |u|^{-a}u$, with $a \in (0, 1)$.

We will assume that the function f satisfies

(f₁) $f \in C(\mathbb{R})$.

(f₂) (i) There exist $\beta^- < 0 < \beta^+$ such that if we set $F(s) = \int_0^s f(t)dt$, it holds that $F(s) < 0$ for all $s \in (\beta^-, \beta^+) \setminus \{0\}$, $F(\beta^\pm) = 0$ and $f(s) > 0$ for all $s \in (\beta^+, \gamma^+)$ and $f(s) < 0$ for all $s \in (\gamma^-, \beta^-)$, where

$$\gamma^+ = \min\{s > \beta^+ \mid f(s) = 0\}, \quad \gamma^+ = \infty \text{ if } f(s) > 0 \text{ for all } s > \beta^+,$$

$$\gamma^- = \max\{s < \beta^- \mid f(s) = 0\}, \quad \gamma^- = -\infty \text{ if } f(s) < 0 \text{ for all } s < \beta^-,$$

(ii) $\lim_{s \rightarrow \gamma^-} F(s) = L = \lim_{s \rightarrow \gamma^+} F(s)$ (L may be finite or not).

(iii) f is locally Lipschitz in $(\gamma^-, 0) \cup (0, \gamma^+)$.

(f₃) Let $Q(s) := mNF(s) - (N - m)sf(s)$.

(a) If $\gamma^+ = \infty$ we assume

(i) There exists $\bar{\beta} > \beta^+$ such that $Q(s) \geq 0$ for all $s \geq \bar{\beta}$,

(ii) There exists $\theta \in (0, 1)$ such that

$$\lim_{s \rightarrow \infty} \left(\inf_{s_1, s_2 \in [\theta s, s]} Q(s_2) \left(\frac{|s|^{m-2}s}{f(s_1)} \right)^{N/m} \right) = \infty \quad (1.2)$$

(b) If $\gamma^- = -\infty$ we assume

(i) There exists $-\bar{\beta} < \beta^-$ such that $Q(s) \geq 0$ for all $s \leq -\bar{\beta}$,

(ii) There exists $\theta \in (0, 1)$ such that

$$\lim_{s \rightarrow -\infty} \left(\inf_{s_1, s_2 \in [s, \theta s]} Q(s_2) \left(\frac{|s|^{m-2}s}{f(s_1)} \right)^{N/m} \right) = \infty \quad (1.3)$$

(f₄) (a) If γ^+ is finite, we assume there exists $L_0 > 0$ and $\bar{\beta} > \beta^+$ such that

$$f(s) \leq L_0(\gamma^+ - s)^{m-1} \quad \text{for all } s \in [\bar{\beta}, \gamma^+), \quad (1.4)$$

(b) If γ^- is finite, we assume there exists $L_0 > 0$ and $\bar{\beta} > -\beta^-$ such that

$$-f(s) \leq L_0(s - \gamma^-)^{m-1} \quad \text{for all } s \in (\gamma^-, -\bar{\beta}]. \quad (1.5)$$

We note that condition (1.2) is stronger than the one used in [GST], where only the limsup is involved. We note that a similar condition was used first by Castro and Kurepa in [CK] for the Laplace operator and later modified in [GHMS] and [GHMZ], where it was used to establish the existence of infinitely many solutions to the Dirichlet problem in a ball for a general quasilinear operator.

The uniqueness of the first bound state solution of (1.1) has been exhaustively studied during the last thirty years, see for example the works [Ch-L], [C1], [CEF1], [CEF2], [FLS], [K], [McL], [McLS], [PeS1], [PeS2], [PuS2], [ST]. The uniqueness of higher order bound states was first studied in [T] for a very special nonlinearity in the semilinear case, and later in [CGHY, CGHY2] for more general nonlinearities obeying some growth restrictions in the so called sub-Serrin case. The proofs are made for the semilinear case, but as it is mentioned there, they can be extended to the quasilinear situation $m > 1$.

Our main result is the following:

Theorem 1.1. *Assume that f satisfies (f_1) – (f_4) . Then given any $k \in \mathbb{N}$, there exists a solution u of (P) having exactly k zeros in $(0, \infty)$.*

Remark 1.2. We will see in section 4 that if there exists $s_0 > 0$ such that for $|s| \geq s_0$, $sf(s) \geq 0$, $F(s) \geq 0$, and

$$(SC) \quad \limsup_{|s| \rightarrow \infty} \frac{sf(s)}{F(s)} < m^*,$$

where as usual $m^* = \frac{Nm}{N-m}$, then (f_3) is satisfied.

We will also see that (f_3) is satisfied for a function f such that

$$f(s) = \frac{s^{m^*-1}}{(\log s)^\lambda}, \quad \text{for some } \lambda > m/(N-m) \text{ and } s \text{ large,}$$

see [FG, Theorem 2]. This function **does not** satisfy (SC) above. We also give an example of a nonlinearity for which γ^+ and $-\gamma^-$ are finite, and an example for which $\gamma^+ < \infty$ and $\gamma^- = -\infty$, allowing **any growth rate** for f at $\pm\infty$ in the first case and any growth rate for f at ∞ in the second case.

To our knowledge, our result is new even for the case of the Laplacian, that is $m = 2$.

In order to prove our result, we will study the behavior of the solutions to the initial value problem

$$\begin{aligned} (\phi_m(u'))' + \frac{N-1}{r} \phi_m(u') + f(u) &= 0, \quad r > 0, \quad N \geq m > 1, \\ u(0) &= \alpha, \quad u'(0) = 0, \end{aligned} \tag{1.6}$$

for $\alpha \in (\beta^+, \gamma^+)$. By a solution to (1.6) we mean a C^1 function u such that $\phi_m(u')$ is also C^1 in its domain.

The idea is to take \mathcal{N}_1 as the set of initial values $\alpha > \beta^+$ such that a solution u of (1.6) has at least one simple zero and decompose it as $\mathcal{N}_1 = \mathcal{N}_2 \cup \mathcal{P}_2 \cup \mathcal{G}_2$, where \mathcal{N}_2 is the set of initial values $\alpha \in \mathcal{N}_1$ such that u has at least two simple zeros in $[0, \infty)$, \mathcal{P}_2 is the set of initial values $\alpha \in \mathcal{N}_1$ such that u has exactly one simple zero in $[0, \infty)$ and from some point on, u remains negative and bounded above by a negative constant, and \mathcal{G}_2 is the set of $\alpha \in \mathcal{N}_1$ such that u is a solution to (1.1) having exactly one sign change in $[0, \infty)$. We prove that $\mathcal{N}_1, \mathcal{N}_2, \mathcal{P}_2$ are open and nonempty. As \mathcal{N}_2 ,

\mathcal{P}_2 and \mathcal{G}_2 are disjoint, we must have that \mathcal{G}_2 is also nonempty. We repeat this idea to obtain higher order bound states.

In section 2, we establish some properties of the solutions to (1.6). We restrict its domain to the set of unique extendibility, and define some crucial sets of initial values. Then in section 3 we prove our main result. In some steps of our proof, we adapt to our situation techniques used in [GST]. Finally, in section 4 we give some examples.

2. SOME PROPERTIES OF THE SOLUTIONS OF THE INITIAL VALUE PROBLEM

The aim of this section is to establish several properties of the solutions to the initial value problem (1.6).

It can be seen, see for example [FLS, NS1, NS2], that solutions are defined and unique at least while they remain nonnegative. Also, if a solution reaches the value zero with a nonzero slope, then this solution can be uniquely continued by considering the equation satisfied by its inverse $r = r(s, \alpha)$ in an appropriate neighborhood of such a point, namely

$$\begin{cases} r' = p \\ p' = \frac{N-1}{m-1} \frac{p^2}{r} + |p|^m p f(s) \\ r(0) = r_0 > 0, \quad p(0) = p_0 \neq 0, \end{cases} \quad (2.1)$$

as the right hand side is Lipschitz in the variable (r, p) .

Moreover, by re-writing the equation in (1.6) in the form

$$\begin{cases} u' = \phi_{m'}(v) \\ v' = -\frac{N-1}{r} v - f(u), \\ u(r_0) = u_0 \quad v(r_0) = v_0 \end{cases} \quad (2.2)$$

where $m' = m/(m-1)$, we see that such a solution can be uniquely extended until it reaches a double zero when $m \leq 2$, as in this case the right hand side is Lipschitz in a neighborhood of any point (u_0, v_0) with $u_0 \neq 0$. When $m > 2$ this is not clear because $\phi_{m'}$ is not locally Lipschitz near 0. Nevertheless, this problem can be handled if $f(u(r_0)) \neq 0$: Assume for simplicity that $f(u_0) < 0$. From the second equation in (2.2), if $\delta > 0$ is small enough to have that for $|r - r_0| < \delta$,

$$-\frac{N-1}{r} v - f(u) \geq \frac{1}{2} |f(u_0)|,$$

(which is possible because $v(r_0) = 0$), then

$$v'(r) \geq \frac{1}{2} |f(u_0)|,$$

implying that

$$v(r) \geq \frac{1}{2} |f(u_0)| (r - r_0).$$

Hence, if (u_1, v_1) and (u_2, v_2) are two solutions of (2.2), then for $r > r_0$, from the mean value theorem, and using that $m' - 2 < 0$, we have

$$\begin{aligned} |(u'_1 - u'_2)(r)| &= |(\phi_{m'}(v_1) - \phi_{m'}(v_2))(r)| = (m' - 1)|\xi|^{m'-2}|(v_1 - v_2)(r)| \\ &\leq C(r - r_0)^{m'-2}|(v_1 - v_2)(r)| \end{aligned}$$

for some positive constant C and thus,

$$|(u_1 - u_2)(r)| \leq C(r - r_0)^{m'-1}\|v_1 - v_2\|,$$

where $\|\cdot\|$ represents the usual sup norm in $C[r_0 - \delta, r_0 + \delta]$. Also from the second equation in (2.2), using that f is locally Lipschitz we find that

$$|(v_1 - v_2)(r)| \leq \int_{r_0}^r |v'_1 - v'_2| \leq C\|v_1 - v_2\|(r - r_0) + K\|u_1 - u_2\|(r - r_0)$$

for some positive constant K . Adding up these two last inequalities we have that

$$\|u_1 - u_2\| + \|v_1 - v_2\| \leq C\delta^{m'-1}\|v_1 - v_2\| + (C + K)\delta\|u_1 - u_2\|,$$

so choosing δ small enough we deduce $u_1 = u_2$ and $v_1 = v_2$ and we have unique extendibility.

If $f(u(r_0)) = 0$, there might be a problem.

Let $u(r) = u(r, \alpha)$ be any solution of (1.6) such that it reaches a first point r_0 where $u'(r_0) = 0$ and $f(u(r_0)) = 0$, and set

$$I(r, \alpha) = \frac{|u'(r)|^m}{m'} + F(u(r)). \quad (2.3)$$

A simple calculation yields

$$I'(r, \alpha) = -\frac{(N-1)}{r}|u'(r)|^m, \quad (2.4)$$

and therefore, as $N \geq m > 1$, we have that I is decreasing in r . As $I(r_0, \alpha) = F(u(r_0)) < 0$, $u(\cdot)$ cannot change sign again. By our assumptions, $u(r_0)$ is either a local minimum or a local maximum for F . If it is a local minimum, then the only possibility is that $u(r) \equiv u(r_0)$ for all $r \geq r_0$. Indeed, for $r \geq r_0$,

$$F(u(r)) \leq \frac{|u'(r)|^m}{m'} + F(u(r)) \leq F(u(r_0)),$$

proving the claim. In particular we have uniqueness if f has only one positive zero and only one negative zero.

Definition 2.1. *The domain D of definition of u will be the domain of unique extendibility.*

Remark 2.2. From the discussion above, it follows that $D = (0, D(\alpha))$, where if $D(\alpha) < \infty$, then $D(\alpha)$ is a double zero of u , or $u'(D(\alpha), \alpha) = 0$ and $u(D(\alpha), \alpha)$ is a relative maximum of F (and thus $F(u(D(\alpha), \alpha)) < 0$).

We will denote by $u(\cdot, \alpha)$ such solution. By standard theory of ordinary differential equations, the solution depends continuously on the initial data in any compact subset of its domain of definition, see for example [CL, Theorem 4.3]. In the rest of this article, and without further mention, the constants $\beta^\pm, \gamma^\pm, \bar{\beta}, \theta$ and L_0 will be as defined in our assumptions (f_2) – (f_4) .

Proposition 2.3. *Let f satisfy (f_1) – (f_2) and let $u(\cdot, \alpha)$ be a solution of (1.6).*

- (i) *There exists $C(\alpha) > 0$ such that $|u(r, \alpha)| \leq C(\alpha)$.*
- (ii) *If $u(\cdot, \alpha)$ is defined in $[0, \infty)$ and $\lim_{r \rightarrow \infty} u(r, \alpha) = \ell$, then*

$$\lim_{r \rightarrow \infty} u'(r, \alpha) = 0 \quad \text{and} \quad \ell \text{ is a zero of } f.$$

Proof. Let $u(r) = u(r, \alpha)$ be any solution of (1.6). From (2.3) and (2.4), and noting that from (f_2) F is bounded below by

$$-\bar{F} = \min_{s \in [\beta^-, \beta^+]} F(s), \quad \bar{F} > 0, \quad (2.5)$$

we have that

$$F(\alpha) \geq F(u(r)) \geq -\bar{F}$$

and thus (i) follows from $(f_2)(ii)$ and the fact that from $(f_2)(i)$, F is strictly increasing in (β^+, γ^+) and strictly decreasing in (γ^-, β^-) .

Assume next $\lim_{r \rightarrow \infty} u(r) = \ell$. Then from (i) ℓ is finite, and as $I(\cdot, \alpha)$ is decreasing and bounded below by $-\bar{F}$, we get that $\lim_{r \rightarrow \infty} u'(r)$ exists. As u' is integrable, we must have $\lim_{r \rightarrow \infty} u'(r) = 0$. Moreover, from the equation in (1.1) and applying L'Hôpital's rule twice, we conclude that

$$\begin{aligned} 0 = \lim_{r \rightarrow \infty} \frac{u(r) - \ell}{r^{m'}} &= - \lim_{r \rightarrow \infty} \frac{r^{\frac{N-1}{m'-1}} |u'(r)|}{m' r^{\frac{N-1}{m'-1}} r^{m'-1}} \\ &= - \frac{1}{m'} \left(\lim_{r \rightarrow \infty} \frac{r^{N-1} |u'(r)|^{m-1}}{r^N} \right)^{m'-1} \\ &= - \frac{1}{m'} \left(\lim_{r \rightarrow \infty} \frac{r^{N-1} f(u(r))}{N r^{N-1}} \right)^{m'-1} = - \frac{1}{m'} \left(\frac{f(\ell)}{N} \right)^{m'-1}, \end{aligned}$$

and (ii) follows. \square

It can be seen that for $\alpha \in [\beta^+, \gamma^+)$, one has $u(r, \alpha) > 0$ and $u'(r, \alpha) < 0$ for r small enough, and thus we can define the extended real number

$$Z_1(\alpha) := \sup\{r \in (0, D(\alpha)) \mid u(s, \alpha) > 0 \text{ and } u'(s, \alpha) < 0 \text{ for all } s \in (0, r)\}.$$

In what follows, we will denote

$$u(Z_i(\alpha), \alpha) = \lim_{r \uparrow Z_i(\alpha)} u(r, \alpha), \quad u'(Z_i(\alpha), \alpha) = \lim_{r \uparrow Z_i(\alpha)} u'(r, \alpha), \quad u(T_i(\alpha), \alpha) = \lim_{r \uparrow T_i(\alpha)} u(r, \alpha).$$

We set

$$\begin{aligned} \mathcal{N}_1 &= \{\alpha \in [\beta^+, \gamma^+) : u(Z_1(\alpha), \alpha) = 0 \text{ and } u'(Z_1(\alpha), \alpha) < 0\} \\ \mathcal{G}_1 &= \{\alpha \in [\beta^+, \gamma^+) : u(Z_1(\alpha), \alpha) = 0 \text{ and } u'(Z_1(\alpha), \alpha) = 0\} \\ \mathcal{P}_1 &= \{\alpha \in [\beta^+, \gamma^+) : u(Z_1(\alpha), \alpha) > 0\}. \end{aligned}$$

From [GST] $\mathcal{N}_1 \neq \emptyset$. Let

$$\tilde{\mathcal{F}}_2 = \{\alpha \in \mathcal{N}_1 : u'(r, \alpha) \leq 0 \text{ for all } r \in (Z_1(\alpha), D(\alpha))\}.$$

For $\alpha \in \mathcal{N}_1 \setminus \tilde{\mathcal{F}}_2$ we define

$$T_1(\alpha) := \sup\{r \in (Z_1(\alpha), D(\alpha)) : u'(r, \alpha) \leq 0\}, \quad U_1(\alpha) = u(T_1(\alpha), \alpha).$$

Also, for $\alpha \in \mathcal{N}_1 \setminus \tilde{\mathcal{F}}_2$ we can define the extended real number

$$Z_2(\alpha) := \sup\{r \in (T_1(\alpha), D(\alpha)) \mid u(s, \alpha) < 0 \text{ and } u'(s, \alpha) > 0 \text{ for all } s \in (T_1(\alpha), r)\}.$$

Let now

$$\mathcal{F}_2 = \{\alpha \in \mathcal{N}_1 \setminus \tilde{\mathcal{F}}_2 : u(Z_2(\alpha), \alpha) < 0\},$$

$$\mathcal{N}_2 = \{\alpha \in \mathcal{N}_1 \setminus \tilde{\mathcal{F}}_2 : u(Z_2(\alpha), \alpha) = 0 \text{ and } u'(Z_2(\alpha), \alpha) > 0\},$$

$$\mathcal{G}_2 = \{\alpha \in \mathcal{N}_1 \setminus \tilde{\mathcal{F}}_2 : u(Z_2(\alpha), \alpha) = 0 \text{ and } u'(Z_2(\alpha), \alpha) = 0\},$$

$$\mathcal{P}_2 = \tilde{\mathcal{F}}_2 \cup \mathcal{F}_2.$$

For $k \geq 3$, and if $\mathcal{N}_{k-1} \neq \emptyset$, we set

$$\tilde{\mathcal{F}}_k = \{\alpha \in \mathcal{N}_{k-1} : (-1)^k u'(r, \alpha) \leq 0 \text{ for all } r \in (Z_{k-1}(\alpha), D(\alpha))\}.$$

For $\alpha \in \mathcal{N}_{k-1} \setminus \tilde{\mathcal{F}}_k$, we set

$$T_{k-1}(\alpha) := \sup\{r \in (Z_{k-1}(\alpha), D(\alpha)) : (-1)^k u'(r, \alpha) \leq 0\},$$

$$U_{k-1}(\alpha) := u(T_{k-1}(\alpha), \alpha).$$

Next, for $\alpha \in \mathcal{N}_{k-1} \setminus \tilde{\mathcal{F}}_k$, we define the extended real number

$$Z_k(\alpha) := \sup\{r \in (T_{k-1}(\alpha), D(\alpha)) \mid (-1)^k u(s, \alpha) < 0 \text{ and } (-1)^k u'(s, \alpha) > 0 \text{ for all } s \in (T_{k-1}(\alpha), r)\}.$$

Finally we set

$$\mathcal{F}_k = \{\alpha \in \mathcal{N}_{k-1} \setminus \tilde{\mathcal{F}}_k : (-1)^k u(Z_k(\alpha), \alpha) < 0\},$$

$$\mathcal{N}_k = \{\alpha \in \mathcal{N}_{k-1} \setminus \tilde{\mathcal{F}}_k : u(Z_k(\alpha), \alpha) = 0 \text{ and } (-1)^k u'(Z_k(\alpha), \alpha) > 0\},$$

$$\mathcal{G}_k = \{\alpha \in \mathcal{N}_{k-1} \setminus \tilde{\mathcal{F}}_k : u(Z_k(\alpha), \alpha) = 0 \text{ and } u'(Z_k(\alpha), \alpha) = 0\},$$

$$\mathcal{P}_k = \tilde{\mathcal{F}}_k \cup \mathcal{F}_k.$$

Concerning the sets \mathcal{N}_k and \mathcal{P}_k we have:

Proposition 2.4. *The sets \mathcal{N}_k and \mathcal{P}_k are open in $[\beta^+, \gamma^+)$ and $\beta^+ \in \mathcal{P}_1$.*

Proof. The proof that \mathcal{N}_k is open is by continuity. Indeed, let $\bar{\alpha} \in \mathcal{N}_k$. We may assume that $u(r, \bar{\alpha})$ is decreasing in $(T_{k-1}(\bar{\alpha}), Z_1(\bar{\alpha}))$, $u(Z_k(\bar{\alpha}), \bar{\alpha}) = 0$ and $u'(Z_k(\bar{\alpha}), \bar{\alpha}) < 0$ (If $k = 1$, we set $T_0(\bar{\alpha}) = 0$).

Since $u'(Z_1(\bar{\alpha}), \bar{\alpha}) < 0$ we can extend the solution $u(r, \bar{\alpha})$ to an interval

$$[T_{k-1}(\bar{\alpha}), Z_k(\bar{\alpha}) + \varepsilon]$$

and $u'(r, \bar{\alpha}) < 0$ for all $r \in (T_{k-1}(\bar{\alpha}), Z_k(\bar{\alpha}) + \varepsilon]$ and the result follows by continuous dependence of the solution on the initial condition.

The proof that $\beta^+ \in \mathcal{P}_1$ and \mathcal{P}_1 is open in $[\beta^+, \gamma^+)$ can be found in [GST]. Let $k > 1$. The proof that \mathcal{P}_k is open is based in the fact that the functional I defined in (2.3) is decreasing in r , and $\alpha \in \mathcal{P}_k$ if and only if $\alpha \in \mathcal{N}_{k-1}$ and $I(r_1, \alpha) < 0$ for some $r_1 \in (0, Z_k(\alpha))$. Hence the openness of \mathcal{P}_k follows from the continuous dependence of solutions to (1.6) in the initial value α and from the openness of \mathcal{N}_{k-1} .

Let $\bar{\alpha} \in \mathcal{P}_k$ and assume first that $Z_k(\bar{\alpha}) = \infty$. From Proposition 2.3,

$$\lim_{r \rightarrow \infty} I(r, \bar{\alpha}) = F(\ell) < 0.$$

Assume next $Z_k(\bar{\alpha}) < \infty$ and $Z_k(\bar{\alpha}) = T_{k-1}(\bar{\alpha})$. Then $I(Z_k(\bar{\alpha}), \bar{\alpha}) < 0$, see Remark 2.2.

The last possibility is $Z_k(\bar{\alpha}) < \infty$ and $Z_k(\bar{\alpha})$ is a maximum point for u with $u(Z_k(\bar{\alpha}), \bar{\alpha}) < 0$, or a minimum point of u with $u(Z_k(\bar{\alpha}), \bar{\alpha}) > 0$, implying that either

$$0 \leq -(\phi_m(u'))'(Z_k(\bar{\alpha}), \bar{\alpha}) = f(u(Z_k(\bar{\alpha}), \bar{\alpha}))$$

and hence $\beta^- < u(Z_k(\bar{\alpha}), \bar{\alpha}) < 0$, or

$$0 \geq -(\phi_m(u'))'(Z_k(\bar{\alpha}), \bar{\alpha}) = f(u(Z_k(\bar{\alpha}), \bar{\alpha}))$$

and thus $0 < u(Z_k(\bar{\alpha}), \bar{\alpha}) < \beta^+$. Therefore in both cases

$$I(Z_k(\bar{\alpha}), \bar{\alpha}) = F(u(Z_k(\bar{\alpha}), \bar{\alpha})) < 0.$$

Conversely, if $\alpha \notin \mathcal{P}_k$ and $\alpha \in \mathcal{N}_{k-1}$, then $\alpha \in \mathcal{G}_k \cup \mathcal{N}_k$, and thus $I(r, \alpha) \geq I(Z_k(\alpha), \alpha) \geq 0$ for all $r \in (0, Z_k(\alpha))$. □

The following lemma is an extension of [GST, Lemma 3.1].

Lemma 2.5. *Let $\alpha \in \mathcal{N}_k$, $T_k(\alpha) < \infty$, and $U_k(\alpha) \in (\bar{\beta}, \gamma^+)$ ($U_k(\alpha) \in (\gamma^-, -\bar{\beta})$). Let $\bar{r} \geq T_k(\alpha)$ be the first point after $T_k(\alpha)$ at which $u(\bar{r}, \alpha) = \bar{\beta}$ ($u(\bar{r}, \alpha) = -\bar{\beta}$). If*

$$\bar{r} \geq \bar{C} := (N-1)(m')^{1/m'} \bar{\beta} \max \left\{ \frac{(\bar{F} + F(\bar{\beta}))^{1/m'}}{F(\bar{\beta})}, \frac{(\bar{F} + F(-\bar{\beta}))^{1/m'}}{F(-\bar{\beta})} \right\}, \quad (2.6)$$

then $\alpha \in \mathcal{N}_{k+1}$.

Proof. The proof of this result is exactly the same as that of Lemma 3.1 in [GST], with $b = \bar{\beta}$, and $R = Z_{k+1}$ and thus we omit it. □

Finally in this section we state a useful and well known Pohozaev type identity (see [P]) which plays a key role in our proofs. For a solution $u(\cdot, \alpha)$ of (1.6), set

$$E(r, \alpha) := mr^N I(r, \alpha) + (N-m)r^{N-1} \phi_m(u'(r, \alpha))u(r, \alpha).$$

Then

$$E'(r, \alpha) = r^{N-1} Q(u(r, \alpha)), \quad (2.7)$$

and thus, in any interval $[R_1, R_2]$ where $uu' \leq 0$ and $u'(R_1, \alpha) = 0$ it holds that

$$mr^N I(r, \alpha) - mR_1^{N-1} F(u(R_1, \alpha)) \geq \int_{R_1}^r t^{N-1} Q(u(t, \alpha)) dt. \quad (2.8)$$

3. EXISTENCE OF BOUND STATES, PROOF OF THEOREM 1.1

The proof of our main result is based in the two crucial lemmas below.

Lemma 3.1. *Assume that f satisfies (f_1) – (f_4) . Then, for each $k \in \mathbb{N}$, there exists $\alpha_k \in (\beta^+, \gamma^+)$ with $\alpha_k < \alpha_{k+1}$ and $[\alpha_k, \gamma^+) \subset \mathcal{N}_k$.*

Proof. If $\gamma^+ < \infty$, then the result for $k = 1$ follows from the proof of Claim 3 case (C1), in [GST, Theorem 1] Moreover, in this case the authors prove that $(f_4)(a)$ implies that

$$\lim_{\alpha \rightarrow \gamma^+} r(\bar{\beta}, \alpha) = \infty,$$

where $r(\bar{\beta}, \alpha)$ is the first value of $r > 0$ such that $u(r, \alpha) = \bar{\beta}$.

In the case $\gamma^+ = \infty$, the result for $k = 1$ follows due to our stronger assumption $(f_3)(a)(ii)$. Indeed, in the proof of Theorem 1, Claim 3 in [GST], the authors prove that if for some $\theta \in (0, 1)$,

$$\inf_{s_1, s_2, \alpha} Q(s_2) \left(\frac{|\alpha|^{m-2} \alpha}{f(s_1)} \right)^{N/m} \geq C_0, \quad \forall s_1, s_2 \in [\theta\alpha, \alpha]$$

where $C_0 > 0$ is a fixed constant, then $\alpha \in \mathcal{N}_1$. ($\mathcal{N}_1 = I^-$ in their notation). By our assumption $(f_3)(ii)$, there exists $\alpha_1 > 0$ such that for all $\alpha \geq \alpha_1$ we have

$$\inf_{s_1, s_2, \alpha} Q(s_2) \left(\frac{|\alpha|^{m-2} \alpha}{f(s_1)} \right)^{N/m} \geq C_0, \quad \forall s_1, s_2 \in [\theta\alpha, \alpha]$$

and thus $[\alpha_1, \infty) \subset \mathcal{N}_1$. Clearly, we can assume that $\alpha_1 \geq \bar{\beta}$.

Let now $k \geq 2$. We will prove our result inductively, starting with $\alpha_1 > \beta^+$ such that $[\alpha_1, \gamma^+) \subset \mathcal{N}_1$.

Claim 1. There exists $\bar{\alpha} \geq \alpha_1$, such that $T_1(\alpha) < \infty$ for all $\alpha \in [\bar{\alpha}, \gamma^+)$.

Assume that this is not true, and hence there exists a sequence $\{\alpha_i\}_{i \geq 2}$, $\alpha_i \rightarrow \gamma^+$ as $i \rightarrow \infty$, such $T_1(\alpha_i) = \infty$ for all $i \geq 2$. Then $u(\cdot, \alpha_i)$ is decreasing, implying, see Proposition 2.3, that $\lim_{r \rightarrow \infty} u(r, \alpha_i) = \ell_i$, $|\ell_i| < |\beta^-|$, see Figure 1.

As

$$|u'(r, \alpha_i)|^m = |u'(r, \alpha_i)|^m + m'F(u(r, \alpha_i)) - m'F(u(r, \alpha_i)) \leq m'I(r, \alpha_i) + m'\bar{F},$$

where \bar{F} is defined in (2.5), we have that

$$\sup_{r \geq r_i} |u'(r, \alpha_i)| \leq m'^{1/m} (I(r, \alpha_i) + \bar{F})^{1/m},$$

and as $u'(r, \alpha_i) \rightarrow 0$ as $r \rightarrow \infty$, $I(r, \alpha_i) \rightarrow L_i = F(\ell_i) < 0$ with $|L_i| \leq \bar{F}$ as $r \rightarrow \infty$.

Denote by $r_i := r(\bar{\beta}, \alpha_i)$ the first value of r where u takes the value $\bar{\beta}$. By integrating (2.4) over $[r_i, \infty)$, we obtain

$$\begin{aligned} I(r_i, \alpha_i) - L_i &= (N-1) \int_{r_i}^{\infty} \frac{|u'(r, \alpha_i)|^m}{r} dr \\ &\leq (N-1)(m')^{1/m'} \frac{(I(r_i, \alpha_i) + \bar{F})^{1/m'}}{r_i} (|\ell_i| + \bar{\beta}) \end{aligned} \quad (3.1)$$

implying that

$$\frac{r_i(I(r_i, \alpha_i) - L_i)}{(I(r_i, \alpha_i) + \bar{F})^{1/m'}} \leq (N-1)(m')^{1/m'}(|\beta^-| + \bar{\beta}). \quad (3.2)$$

We will show that (3.2) is not possible.

We have two possibilities: either **(a)** r_i is bounded, or **(b)** $r_i \rightarrow \infty$ through some subsequence. We recall that in case $\gamma^+ < \infty$, only **(b)** is possible.

Case **(a)**. In this case necessarily $\gamma^+ = \infty$. We assume that there is $M > 0$ such that $r_i \leq M$ for all i .

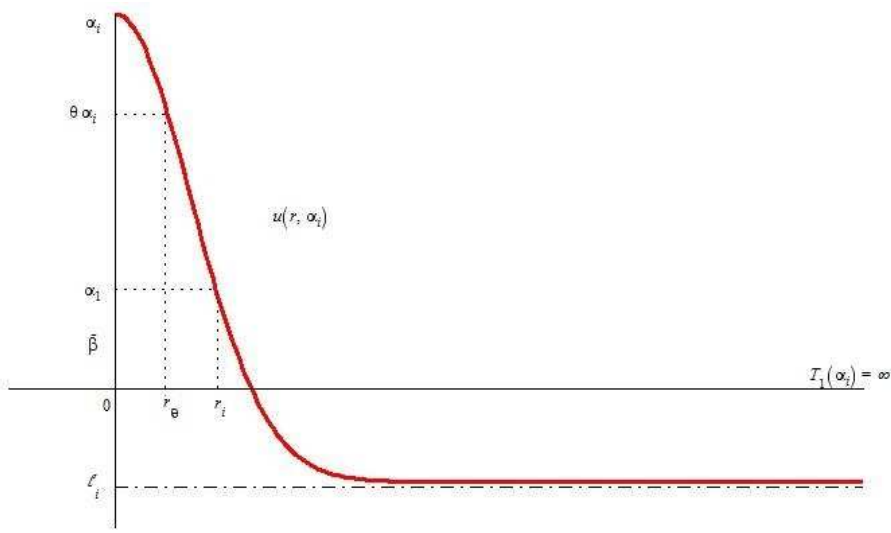


FIGURE 1. The shape of a solution satisfying $T_1(\alpha_i) = \infty$

Let $\theta \in (0, 1)$ be as in assumption $(f_3)(ii)$ and let i be large enough to have $\theta\alpha_i > \bar{\beta}$. By setting $r_\theta > 0$ the point where $u(r_\theta, \alpha_i) = \theta\alpha_i$, integration of (2.7) over $[0, r_i]$ yields

$$\begin{aligned} mr_i^N I(r_i, \alpha_i) &\geq \left(\int_0^{r_\theta} + \int_{r_\theta}^{r_i} \right) t^{N-1} Q(u(t, \alpha_i)) dt \\ &\geq \int_0^{r_\theta} t^{N-1} Q(u(t, \alpha_i)) dt \quad (\text{as } Q(u(t, \alpha_i)) \geq 0 \text{ in } [r_\theta, r_i]) \\ &\geq Q(s_2) \frac{r_\theta^N}{N} \quad \text{where we have set } Q(s_2) = \min_{s \in [\theta\alpha_i, \alpha_i]} Q(s). \end{aligned} \quad (3.3)$$

Now we estimate r_θ : Set $f(s_1) = \max_{s \in [\theta\alpha_i, \alpha_i]} f(s)$ ($s_1 \in [\theta\alpha_i, \alpha_i]$). From the equation in (1.6),

$$-r^{N-1} \phi_m(u'(r, \alpha_i)) = \int_0^r t^{N-1} f(u(t, \alpha_i)) dt \leq f(s_1) \frac{r^N}{N},$$

hence

$$-u'(r, \alpha_i) \leq \phi_{m'}(f(s_1)) \frac{r^{m'-1}}{N^{m'-1}}.$$

Integrating now this last inequality over $[0, r_\theta]$ we obtain

$$(1 - \theta)\alpha_i \leq \phi_{m'}(f(s_1)) \frac{r_\theta^{m'}}{m' N^{m'-1}}$$

and thus

$$r_\theta \geq \left(\frac{c\alpha_i^{m-1}}{f(s_1)} \right)^{1/m},$$

where $c = ((1 - \theta)m')^{m-1}N$. Therefore, by $(f_3)(ii)$ we conclude that

$$mr_i^N I(r_i, \alpha_i) \geq \frac{1}{N} Q(s_2) \left(\frac{c\alpha_i^{m-1}}{f(s_1)} \right)^{N/m} \rightarrow \infty \text{ as } i \rightarrow \infty,$$

hence, as $N \geq m$ and by the boundedness assumption on r_i , we obtain that

$$r_i^m I(r_i, \alpha_i) \rightarrow \infty \text{ as } i \rightarrow \infty$$

contradicting (3.2).

Case **(b)**. From (3.2),

$$0 \leq \frac{I(r_i, \alpha_i) - L_i}{(I(r_i, \alpha_i) + \bar{F})^{1/m'}} \leq \frac{N-1}{r_i} (m')^{1/m'} (|\beta^-| + \bar{\beta}). \quad (3.4)$$

We note first that in this case $I(r_i, \alpha_i) \rightarrow \infty$ as $i \rightarrow \infty$. Indeed, if for some subsequence (still renamed the same) we have that $I(r_i, \alpha_i)$ is bounded, then from (3.4) it must be that

$$I(r_i, \alpha_i) - L_i \rightarrow 0 \text{ as } i \rightarrow \infty,$$

which is a contradiction because

$$I(r_i, \alpha_i) - L_i \geq F(\bar{\beta}) - L_i \geq F(\bar{\beta}) > 0.$$

Hence $I(r_i, \alpha_i) \rightarrow \infty$ as $i \rightarrow \infty$. Then as

$$\frac{I(r_i, \alpha_i) - L_i}{(I(r_i, \alpha_i) + \bar{F})^{1/m'}} = (I(r_i, \alpha_i) + \bar{F})^{1/m} - \frac{\bar{F} + L_i}{(I(r_i, \alpha_i) + \bar{F})^{1/m'}},$$

we get from (3.4) that

$$\bar{F}^{1/m} < (I(r_i, \alpha_i) + \bar{F})^{1/m} \leq \frac{\bar{F} + L_i}{(I(r_i, \alpha_i) + \bar{F})^{1/m'}} + \frac{N-1}{r_i} (m')^{1/m'} (|\beta^-| + \bar{\beta}) \rightarrow 0,$$

also a contradiction and our Claim 1 follows.

Claim 2. $\lim_{\alpha \rightarrow \gamma^+} u(T_1(\alpha), \alpha) = \gamma^-$.

We will argue by contradiction and thus assume that there exists $\gamma^- < M < \beta^-$ and a sequence $\{\alpha_i\} \rightarrow \gamma^+$ such that $U_1(\alpha_i) = u(T_1(\alpha_i), \alpha_i) \geq M$. From Claim 1 we may assume that $T_1(\alpha_i) < \infty$ for all i .

Let us denote by $r(\cdot, \alpha_i)$ the inverse of $u(\cdot, \alpha_i)$ in the interval $[0, T_1(\alpha_i)]$. From $(f_2)(i), (ii)$, there exists $\tilde{M} \in (\beta^+, \gamma^+)$ such that $F(\tilde{M}) - F(M) > 0$. Integrating now I' over $[r(\tilde{M}, \alpha_i), T_1(\alpha_i)]$ we obtain

$$\begin{aligned} I(r(\tilde{M}, \alpha_i), \alpha_i) - F(U_1(\alpha_i)) &= (N-1) \int_{r(\tilde{M}, \alpha_i)}^{T_1(\alpha_i)} \frac{|u'(r, \alpha_i)|^m}{r} dr \\ &\leq (N-1)(m')^{\frac{1}{m'}} \frac{(I(r(\tilde{M}, \alpha_i), \alpha_i) + \bar{F})^{\frac{1}{m'}}}{r(\tilde{M}, \alpha_i)} (|M| + \tilde{M}), \end{aligned}$$

hence

$$I(r(\tilde{M}, \alpha_i), \alpha_i) - F(M) \leq (N-1)(m')^{\frac{1}{m'}} \frac{(I(r(\tilde{M}, \alpha_i), \alpha_i) + \bar{F})^{\frac{1}{m'}}}{r(\tilde{M}, \alpha_i)} (|M| + \tilde{M}).$$

We can prove that this inequality is not possible by using the same arguments we used to prove that (3.2) led to a contradiction, hence our claim follows.

Claim 3. There exists $\alpha_2 > \alpha_1$ such that $[\alpha_2, \gamma^+) \subset \mathcal{N}_2$.

We argue by contradiction assuming that there exists a sequence $\{\alpha_i\} \rightarrow \gamma^+$ such that $\alpha_i \in \mathcal{G}_2 \cup \mathcal{P}_2$ for all i and study the solutions $u(\cdot, \alpha_i)$ in the interval $(0, Z_2(\alpha_i))$. Our argument follows the ideas of the proof of [GST, Theorem 1].

From Claim 1 we may assume that $T_1 = T_1(\alpha_i) < \infty$.

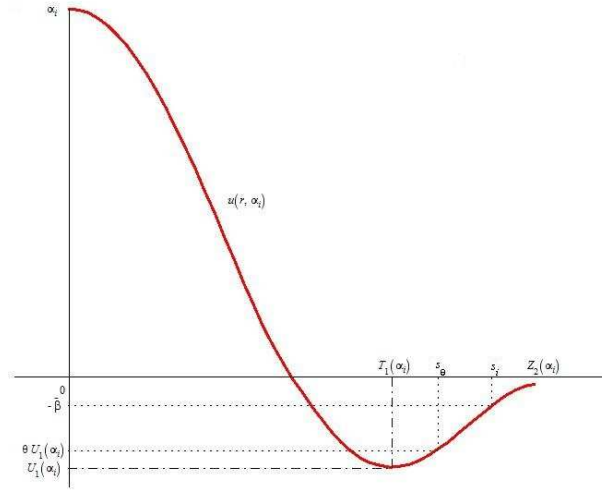
Case $\gamma^- = -\infty$: In this case from Claim 2, $U_1(\alpha_i) \rightarrow -\infty$ as $i \rightarrow \infty$. Also, by (f_1) and $(f_3)(b)(i)$, Q is bounded below, and as $Q(\beta^-) = -(N-m)\beta^- f(\beta^-) < 0$, we can set

$$0 < \bar{Q} = -\inf_{s \leq 0} Q(s).$$

Let $s_\theta(\alpha_i) > T_1(\alpha_i)$ be now the point where $u(s_\theta, \alpha_i) = \theta U_1(\alpha_i)$, where by Claim 2 we may assume that $-\theta U_1(\alpha_i) > \bar{\beta}$ for all i , and let $s_i > T_1(\alpha_i)$ be the first point after $T_1(\alpha_i)$ where $u(s_i, \alpha_i) = -\bar{\beta}$ (and thus from Lemma 2.5, $s_i < \bar{C}$), see Figure 2 below.

Then, for any $r \in (T_1(\alpha_i), Z_2(\alpha_i))$, by (2.8) we have

$$\begin{aligned} mr^N I(r, \alpha_i) - mT_1^N F(U_1(\alpha_i)) &\geq \left(\int_{T_1}^{s_\theta} + \underbrace{\int_{s_\theta}^{s_i}}_{\geq 0} + \int_{s_i}^r \right) t^{N-1} Q(u(t, \alpha_i)) dt \\ &\geq \int_{T_1}^{s_\theta} t^{N-1} Q(u(t, \alpha_i)) dt - \bar{Q} \frac{r^N}{N} \\ &\geq Q(s_2) \left(\frac{s_\theta^N - T_1^N}{N} \right) - \bar{Q} \frac{r^N}{N} \\ &\geq Q(s_2) \frac{s_\theta^N}{N} - T_1^N \frac{NmF(s_2)}{N} - \bar{Q} \frac{r^N}{N} \end{aligned}$$

FIGURE 2. The shape of a solution such that $\alpha_i \in \mathcal{G}_2 \cup \mathcal{P}_2$

for some $s_2 \in [U_1(\alpha_i), \theta U_1(\alpha_i)]$. s_θ can now be estimated as we did before but setting now $f(s_1) = \min_{s \in [U_1(\alpha_i), \theta U_1(\alpha_i)]} f(s)$ ($s_1 \in [U_1(\alpha_i), \theta U_1(\alpha_i)]$). We obtain that

$$s_\theta \geq \left(\frac{c|U_1(\alpha_i)|^{m-1}}{|f(s_1)|} \right)^{1/m}, \quad (3.5)$$

where $c = ((1 - \theta)m')^{m-1}N$. Therefore, for $r \in (T_1(\alpha_i), Z_2(\alpha_i))$ we have

$$\begin{aligned} mr^N I(r, \alpha_i) &\geq Q(s_2) \frac{s_\theta^N}{N} - T_1^N \frac{NmF(s_2)}{N} + mT_1^N F(U_1(\alpha_i)) - \bar{Q} \frac{r^N}{N} \\ &\geq \frac{1}{N} Q(s_2) \left(\frac{c|U_1(\alpha_i)|^{m-1}}{|f(s_1)|} \right)^{N/m} - mT_1^N \underbrace{(F(s_2) - F(U_1(\alpha_i)))}_{\leq 0} - \bar{Q} \frac{r^N}{N} \\ &\geq \frac{1}{N} Q(s_2) \left(\frac{c|U_1(\alpha_i)|^{m-1}}{|f(s_1)|} \right)^{N/m} - \bar{Q} \frac{r^N}{N}. \end{aligned} \quad (3.6)$$

As in [GST], for an arbitrary $a > 0$ we set $R_a = \min\{\bar{C} + a, Z_2(\alpha_i)\}$. From Lemma 2.5, $R_a \in (s_i, Z_2(\alpha_i))$. Also, by definition $R_a \leq \bar{C} + a$ which is finite and independent of α_i , hence, by $(f_3)(ii)$, we can choose i_0

$$I(r, \alpha_i) \geq F(-\bar{\beta}) + \frac{1}{m'} \left(\frac{\bar{\beta}}{a} \right)^m \quad \text{for all } r \in (s_i, R_a) \quad \text{and } i \geq i_0. \quad (3.7)$$

As in [GST], we prove that for these α_i , it holds that $R_a = \bar{C} + a$. Indeed, if not, $R_a = Z_2(\alpha_i)$ and, as $\alpha_i \in \mathcal{G}_2 \cup \mathcal{P}_2$, it must be that $u'(R_a, \alpha_i) = 0$, implying that $I(R_a, \alpha_i) = F(u(R_a, \alpha_i)) < F(-\bar{\beta})$ contradicting (3.7). But, by (3.7), $|u'(r, \alpha_i)| \geq \bar{\beta}/a$ in (s_i, R_a) , hence

$$u(R_a, \alpha_i) - u(s_i, \alpha_i) \geq \frac{\bar{\beta}}{a} (R_a - s_i)$$

implying

$$\begin{aligned} u(R_a, \alpha_i) &\geq -\bar{\beta} + \frac{\bar{\beta}}{a}(\bar{C} + a - s_i) \\ &= \frac{\bar{\beta}}{a}(\bar{C} - s_i) > 0, \end{aligned}$$

a contradiction. Hence there exists $\alpha_2 > \alpha_1$ such that $[\alpha_2, \gamma^+) \subset \mathcal{N}_2$.

Case $\gamma^- > -\infty$:

In this case from Claim 2, $U_1(\alpha_i) \downarrow \gamma^-$. By $(f_4)(b)$, there exists $\bar{\beta} > 0$ with $\gamma^- < -\bar{\beta} < \beta^-$, such that

$$\frac{-f(s)}{(s - \gamma^-)^{m-1}} < L_0 \quad \text{for all } s \in (\gamma^-, -\bar{\beta}].$$

Let now $i_0 \in \mathbb{N}$ be such that

$$\gamma^- < U_1(\alpha_i) < -\bar{\beta} \quad \text{for all } i \geq i_0.$$

Denote again by s_i the first point after $T_1(\alpha_i)$ where $u(s_i, \alpha_i) = -\bar{\beta}$. By integrating the equation in (1.6) over $[T_1(\alpha_i), r]$, with $r \in (T_1(\alpha_i), s_i]$, we obtain

$$\begin{aligned} r^{N-1} \phi_m(u'(r, \alpha_i)) &\leq L_0 \int_{T_1(\alpha_i)}^r t^{N-1} (u(t, \alpha_i) - \gamma^-)^{m-1} dt \\ &\leq \frac{L_0}{N} (u(r, \alpha_i) - \gamma^-)^{m-1} r^N, \end{aligned}$$

hence

$$\begin{aligned} 0 \leq u'(r, \alpha_i) &\leq C(u(r, \alpha_i) - \gamma^-) s_i^{1/(m-1)} \\ &= C(u(T_1(\alpha_i), \alpha_i) + \int_{T_1(\alpha_i)}^r u'(t, \alpha_i) dt - \gamma^-) s_i^{1/(m-1)}, \end{aligned}$$

where $C = (L_0/N)^{m'-1}$. Using Gronwall's inequality we get

$$u'(r, \alpha_i) \leq C s_i^{1/(m-1)} (u(T_1(\alpha_i), \alpha_i) - \gamma^-) \exp(C s_i^{m/(m-1)}),$$

implying as in [GST] that

$$\lim_{i \rightarrow \infty} s_i = \infty. \tag{3.8}$$

Using now Lemma 2.5 we conclude that $\alpha_i \in \mathcal{N}_2$ for i large enough, which contradicts our initial assumption. Hence there exists $\alpha_2 > \alpha_1$ such that $[\alpha_2, \gamma^+) \subset \mathcal{N}_2$.

Claim 4. There exists $\bar{\alpha} > \alpha_2$ such that $T_2(\alpha) < \infty$ for all $\alpha \in [\bar{\alpha}, \gamma^+)$.

Assume as in the proof of Claim 1 that this is not true, and hence there exists a sequence $\{\alpha_i\}_{i \geq 3}$, $\alpha_i \rightarrow \gamma^+$ as $i \rightarrow \infty$, such $T_2(\alpha_i) = \infty$ for all $i \geq 3$. Then $u(\cdot, \alpha_i)$ is increasing, implying, see Proposition 2.3, that $\lim_{r \rightarrow \infty} u(r, \alpha_i) = \ell_i$, $\ell_i < \beta^+$.

Following the proof in Claim 1, we denote once more by $s_i := r(-\bar{\beta}, \alpha_i)$ the first value of r after $T_1(\alpha_i)$ where u takes the value $-\bar{\beta}$. By integrating (2.4) over $[s_i, \infty)$, we obtain

$$\frac{s_i(I(s_i, \alpha_i) - L_i)}{(I(s_i, \alpha_i) + \bar{F})^{1/m'}} \leq (m')^{1/m'}(\beta^+ + \bar{\beta}). \quad (3.9)$$

We will show that (3.9) is not possible. Again we have two possibilities: either **(a)** s_i is bounded, or **(b)** $s_i \rightarrow \infty$ through some subsequence. We recall that in case $\gamma^- > -\infty$, only **(b)** is possible, see (3.8). The proof in case **(b)** is the same as the corresponding one in Claim 1. In case **(a)**, some minor changes are needed: the integral in (3.3) changes to

$$\begin{aligned} mr^N I(s_i, \alpha_i) - mT_1^N F(U_1(\alpha_i)) &\geq \left(\int_{T_1}^{s_\theta} + \underbrace{\int_{s_\theta}^{s_i}}_{\geq 0} \right) t^{N-1} Q(u(t, \alpha_i)) dt \\ &\geq \int_{T_1}^{s_\theta} t^{N-1} Q(u(t, \alpha_i)) dt \\ &\geq Q(s_2) \left(\frac{s_\theta^N - T_1^N}{N} \right) \\ &\geq Q(s_2) \frac{s_\theta^N}{N} - T_1^N m F(s_2), \end{aligned}$$

where $s_\theta(\alpha_i) > T_1(\alpha_i)$ is the first point after $T_1(\alpha_i)$ where $u(s_\theta, \alpha_i) = \theta U_1(\alpha_i)$ and $Q(s_2) := \min_{s \in [U_1(\alpha_i), \theta U_1(\alpha_i)]} Q(s)$. As $F(U_1(\alpha_i)) - F(s_2) \geq 0$, we obtain

$$mr^N I(s_i, \alpha_i) \geq Q(s_2) \frac{s_\theta^N}{N},$$

and thus we can continue arguing as in the proof of Claim 1 using the estimate for s_θ given in (3.5).

Proof of the lemma:

We can repeat the arguments in Claim 2, to prove that $U_2(\alpha) = u(T_2(\alpha), \alpha) \rightarrow \gamma^+$ as $\alpha \rightarrow \gamma^+$. Then we argue as in the proof of Claim 3 to show the existence of $\alpha_3 > \alpha_2$ such that $[\alpha_3, \infty) \subset \mathcal{N}_3$, see Figure 3 above. Repeating the argument as many times as necessary the lemma follows. \square

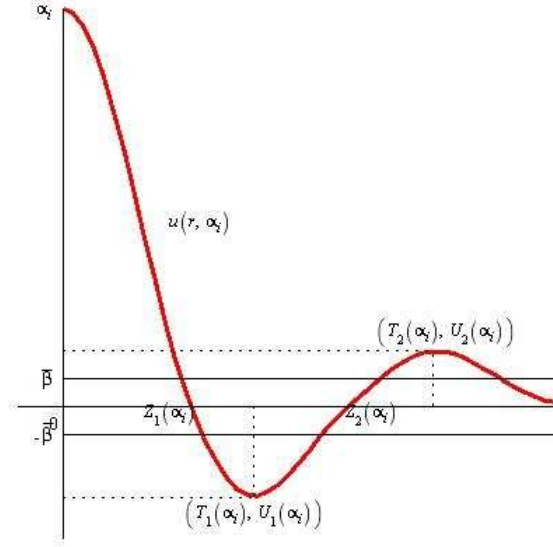
In view of this lemma, the natural candidate to belong to \mathcal{G}_k is

$$\alpha_k^* = \inf\{\alpha \geq \beta^+ \mid [\alpha, \gamma^+) \subseteq \mathcal{N}_k\}.$$

In order to prove that $\alpha_k^* \in \mathcal{G}_k$ we need the following result:

Lemma 3.2. *Assume that f satisfies (f_1) – (f_2) , let $k \in \mathbb{N}$ and let $\bar{\alpha} \in \mathcal{G}_k$. If $\{\alpha_i\} \subset \mathcal{N}_k$ is any sequence such that $\alpha_i \rightarrow \bar{\alpha}$ as $i \rightarrow \infty$, then there exists i_0 such that*

$$\alpha_i \in \mathcal{P}_{k+1} \quad \text{for all } i \geq i_0. \quad (3.10)$$

FIGURE 3. The shape of $u(r, \alpha_i)$ for $\alpha_i \in \mathcal{G}_3 \cup \mathcal{P}_3$.

Proof. Let $\bar{\alpha} \in \mathcal{G}_k$ and let (see Figure 4 below) $\alpha_i \rightarrow \bar{\alpha}$ with $\alpha_i \in \mathcal{N}_k$. Assume by contradiction that $\{\alpha_i\}$ contains a subsequence, still denoted the same, such that $\{\alpha_i\} \subset \mathcal{N}_{k+1} \cup \mathcal{G}_{k+1}$. Then $T_k(\alpha_i) < \infty$ for all i , and since $I(Z_{k+1}(\alpha_i), \alpha_i) \geq 0$, it follows that $I(T_k(\alpha_i), \alpha_i) > 0$. Without loss of generality we may assume that $u(\cdot, \bar{\alpha})$ is decreasing in $(T_{k-1}(\bar{\alpha}), Z_k(\bar{\alpha}))$ so that $T_k(\alpha_i)$ is a minimum point for $u(\cdot, \alpha_i)$ and therefore

$$u(T_k(\alpha_i), \alpha_i) < \beta^-. \quad (3.11)$$

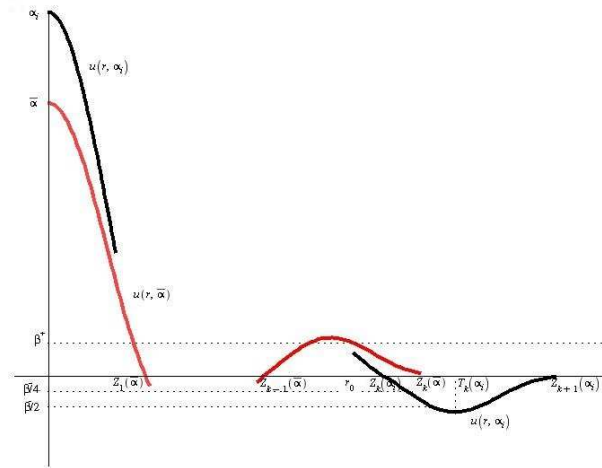


FIGURE 4. Situation in Lemma 3.2

Let us denote by $r(\cdot, \alpha_i)$ the inverse of $u(\cdot, \alpha_i)$ in $(T_{k-1}(\alpha_i), T_k(\alpha_i))$ and let now $\varepsilon \in (0, 1)$. Since

$$\lim_{r \rightarrow Z_k(\bar{\alpha})} I(r, \bar{\alpha}) = 0 \quad \text{and} \quad \lim_{r \rightarrow Z_k(\bar{\alpha})} u(r, \bar{\alpha}) = 0,$$

there exists $r_0 > T_{k-1}(\bar{\alpha})$ such that

$$I(r_0, \bar{\alpha}) < \varepsilon, \quad 0 < u(r_0, \bar{\alpha}) < \beta^+/2,$$

and therefore by continuity, for i large enough, $0 < u(r_0, \alpha_i) < \beta^+$, $Z_k(\alpha_i) > r_0$, and

$$I(r_0, \alpha_i) < 2\varepsilon.$$

Since I is decreasing in r , we have that

$$I(r, \alpha_i) < 2\varepsilon \quad \text{for all } r \in (r_0, Z_{k+1}(\alpha_i)),$$

hence

$$|u'(r, \alpha_i)| \leq \sqrt[m]{\bar{F} + 2} := K \quad \text{for all } r \in (r_0, Z_{k+1}(\alpha_i)) \quad (3.12)$$

and i large enough. From (3.11), $[\beta^-, 0] \subset [u(T_k(\alpha_i), \alpha_i), 0]$, and from (3.12), by the mean value theorem we obtain that

$$r\left(\frac{\beta^-}{2}, \alpha_i\right) - r\left(\frac{\beta^-}{4}, \alpha_i\right) \geq \frac{|\beta^-|}{4K}. \quad (3.13)$$

Let now

$$H(r, \alpha) = r^{m'(N-1)} I(r, \alpha).$$

Then

$$H'(r, \alpha) = m'(N-1)r^{m'(N-1)-1} F(u(r, \alpha)),$$

implying that for $\alpha = \bar{\alpha}$, $H'(r, \bar{\alpha}) < 0$ for all $r \in (r_0, Z_k(\bar{\alpha}))$ and

$$H(r, \bar{\alpha}) \downarrow L \geq 0$$

as $r \rightarrow Z_k(\bar{\alpha})$. Also, by choosing a larger r_0 if necessary, we may assume $H(r_0, \bar{\alpha}) < L + \varepsilon$. Thus by continuity we may assume that

$$H(r_0, \alpha_i) \leq L + 2\varepsilon \quad \text{for } i \text{ large enough.}$$

Also, as $u(r, \alpha_i) < \beta^+$ for $r \in [r_0, Z_k(\alpha_i)]$, H is decreasing in $[r_0, Z_k(\alpha_i)]$ implying

$$H(Z_k(\alpha_i), \alpha_i) \leq L + 2\varepsilon \quad \text{for } i \text{ large enough.}$$

Integrating $H'(\cdot, \alpha_i)$ over $(Z_k(\alpha_i), r(\frac{\beta^-}{2}, \alpha_i))$, we find that

$$H\left(r\left(\frac{\beta^-}{2}, \alpha_i\right), \alpha_i\right) - H(Z_k(\alpha_i), \alpha_i) = -m'(N-1) \int_{Z_k(\alpha_i)}^{r(\frac{\beta^-}{2}, \alpha_i)} t^{m'(N-1)-1} |F(u(t, \alpha_i))| dt$$

and thus, observing that since $N \geq m$, we have $m'(N-1) - 1 \geq m-1 > 0$ implying

$$\begin{aligned}
H\left(r\left(\frac{\beta^-}{2}, \alpha_i\right), \alpha_i\right) &\leq L + 2\varepsilon - m'(N-1)(Z_k(\alpha_i))^{m'(N-1)-1} \int_{Z_k(\alpha_i)}^{r(\frac{\beta^-}{2}, \alpha_i)} |F(u(t, \alpha_i))| dt \\
&\leq L + 2\varepsilon - m'(N-1)(Z_k(\alpha_i))^{m'(N-1)-1} \int_{r(\frac{\beta^-}{4}, \alpha_i)}^{r(\frac{\beta^-}{2}, \alpha_i)} |F(u(t, \alpha_i))| dt \\
&\leq L + 2\varepsilon - m'(N-1)(Z_k(\alpha_i))^{m'(N-1)-1} \left(r\left(\frac{\beta^-}{2}, \alpha_i\right) - r\left(\frac{\beta^-}{4}, \alpha_i\right)\right) C \\
&\leq L + 2\varepsilon - m'(N-1)(Z_k(\alpha_i))^{m'(N-1)-1} \frac{b}{4K} C,
\end{aligned}$$

where $C := \inf\{|F(s)|, s \in [\frac{\beta^-}{2}, \frac{\beta^-}{4}]\}$. If $Z_k(\bar{\alpha}) = \infty$, by taking i larger if necessary, we conclude that $H(r(\frac{\beta^-}{2}, \alpha_i), \alpha_i) < 0$ and thus $\alpha_i \in \mathcal{P}_{k+1}$, a contradiction. If $Z_k(\bar{\alpha}) < \infty$, the same conclusion follows by observing that in this case $L = 0$ and $Z_k(\alpha_i)$ is bounded below by $\bar{r}/2$, where \bar{r} the first value of $r > 0$ where $u(\cdot, \bar{\alpha})$ takes the value β^+ . \square

Remark 3.3. Since $\bar{\alpha} \in \mathcal{G}_k \subset \mathcal{N}_{k-1}$, by Proposition 2.4, there exists $\delta_0 > 0$ such that

$$(\bar{\alpha} - \delta_0, \bar{\alpha} + \delta_0) \subset \mathcal{N}_{k-1} = \mathcal{P}_k \cup \mathcal{G}_k \cup \mathcal{N}_k,$$

therefore, from the previous lemma, there exists $\delta \in (0, \delta_0)$ such that

$$(\bar{\alpha} - \delta, \bar{\alpha} + \delta) \subset \mathcal{P}_k \cup \mathcal{G}_k \cup \mathcal{P}_{k+1}.$$

Proof of Theorem 1.1. From Lemma 3.1, for each $k \in \mathbb{N}$ the set $\{\alpha \geq \beta^+ \mid (\alpha, \gamma^+) \subseteq \mathcal{N}_k\}$ is nonempty and thus we can set

$$\alpha_k^* := \inf\{\alpha \geq \beta^+ \mid (\alpha, \gamma^+) \subseteq \mathcal{N}_k\}.$$

Clearly,

$$(\alpha_k^*, \gamma^+) \subseteq \mathcal{N}_k.$$

We will prove that $\alpha_k^* \in \mathcal{G}_k$ for all k . The result will follow by induction over k .

$\alpha_1^* \in \mathcal{G}_1$: Indeed, if not, then by definition $\alpha_1^* \in \mathcal{N}_1 \cup \mathcal{P}_1$, which cannot be because from Proposition 2.4, the sets \mathcal{N}_1 and \mathcal{P}_1 are open and disjoint and $\beta^+ \in \mathcal{P}_1$.

Assume that the assertion is true for $k = j$, that is, $\alpha_j^* \in \mathcal{G}_j$. As before, by Proposition 2.4, α_{j+1}^* cannot belong to $\mathcal{P}_{j+1} \cup \mathcal{N}_{j+1}$, so in order to prove the result we only need to prove that $\alpha_{j+1}^* \in \mathcal{N}_j$. From Lemma 3.2, there exists $\bar{\alpha} \in (\alpha_j^*, \gamma^+)$ such that $\bar{\alpha} \in \mathcal{P}_{j+1}$. As $(\alpha_{j+1}^*, \gamma^+) \in \mathcal{N}_{j+1}$, it must be that $\bar{\alpha} \leq \alpha_{j+1}^*$, hence $\alpha_j^* < \alpha_{j+1}^*$ and thus $\alpha_{j+1}^* \in \mathcal{N}_j$. \square

4. CONCLUDING REMARKS AND EXAMPLES

We begin this section by discussing Remark 1.2. Assume that condition (SC) holds and s_0 is such that $sf(s) \geq 0$ $F(s) \geq 0$ for $|s| \geq s_0$. Then there exists $\varepsilon > 0$, and

$\bar{s}_0 \geq s_0$ such that for $s \geq \bar{s}_0$,

$$(i) \quad sf(s) \leq (m^* - \varepsilon)F(s) \quad \text{and hence} \quad (ii) \quad \frac{F(s)}{s^{m^* - \varepsilon}} \quad \text{decreases.} \quad (4.1)$$

Let $\theta \in (0, 1)$ be fixed and let $s \geq \bar{s}_0/\theta$. Then there exists a positive constant C so that for any $s_2 \in [\theta s, s]$ we have

$$Q(s_2) = (Nm - (N - m)\frac{s_2 f(s_2)}{F(s_2)})F(s_2) \geq CF(s_2) \geq CF(\theta s).$$

Also, for any $s_1 \in [\theta s, s]$,

$$\frac{s^{m-1}}{f(s_1)} \geq \frac{s_1^{m-1}}{f(s_1)},$$

and thus

$$\begin{aligned} Q(s_2) \left(\frac{s^{m-1}}{f(s_1)} \right)^{N/m} &\geq CF(\theta s) \left(\frac{s_1^{m-1}}{f(s_1)} \right)^{N/m} = CF(\theta s) \left(\frac{s_1^m}{s_1 f(s_1)} \right)^{N/m} \\ &\text{by (4.1) (i)} \geq \bar{C}F(\theta s) \left(\frac{s_1^m}{F(s_1)} \right)^{N/m} \\ &= \bar{C}F(\theta s) \left(\frac{s_1^{m^* - \varepsilon}}{F(s_1)} \right)^{N/m} s_1^{N - (m^* - \varepsilon)N/m} \\ &\text{by (4.1) (ii)} \geq \bar{C}F(\theta s) \left(\frac{(\theta s)^{m^* - \varepsilon}}{F(\theta s)} \right)^{N/m} s^{N - (m^* - \varepsilon)N/m} \\ &= \bar{C} \left(\frac{(\theta s)^{m^* - \varepsilon}}{F(\theta s)} \right)^{(N-m)/m} (\theta s)^{m^* - \varepsilon} s^{N - (m^* - \varepsilon)N/m} \\ &= \bar{C} \theta^{m^* - \varepsilon} \left(\frac{(\theta s)^{m^* - \varepsilon}}{F(\theta s)} \right)^{(N-m)/m} s^{\varepsilon(N-m)/m} \\ &\text{by (4.1) (ii)} \geq C \theta^{m^* - \varepsilon} \left(\frac{(\bar{s}_0)^{m^* - \varepsilon}}{F(\bar{s}_0)} \right)^{(N-m)/m} s^{\varepsilon(N-m)/m}. \end{aligned}$$

Since the argument is the same for $-s$ large, (f_3) is satisfied.

We note that the canonical example $f(s) = |s|^{p-2}s - |s|^{q-2}s$ (which is not Lipschitz at 0 for $q < 2$) satisfies condition (SC) for $1 < q < p < m^*$ and thus our theorem applies to this f .

Next we deal with [FG, Theorem 2], that is we consider f satisfying (f_1) and (f_2) such that for some $s_0 > \max\{2e, 2\beta^+\}$ and $\lambda > \frac{m}{N-m}$, it holds that

$$f(s) = \frac{|s|^{m^* - 2}s}{(\log(|s|))^\lambda}, \quad |s| \geq s_0,$$

where as usual $m^* = Nm/(N - m)$. It can be easily verified that in this case

$$\limsup_{|s| \rightarrow \infty} \frac{sf(s)}{F(s)} = m^*$$

and thus (SC) is not satisfied. We will see that this f satisfies (f_3) and thus problem (1.1) with this f has bound states having any prescribed number of zeros.

We first observe that for $s \geq s_0$, Q is an increasing function. Indeed,

$$Q'(s) = (N(m-1) + m)f(s) - (N-m)sf'(s) = \lambda(N-m)f(s)(\log(s))^{-1} > 0,$$

hence for $s_2 \in (\theta s, s)$, $Q(s_2) \geq Q(\theta s)$. On the other hand, for $s_1 \in [\theta s, s]$,

$$\begin{aligned} \left(\frac{s^{m-1}}{f(s_1)} \right)^{N/m} &= \left(s^{m-1} s_1^{1-m^*} (\log(s_1))^\lambda \right)^{N/m} \\ &\geq \left(s^{m-m^*} (\log(\theta s))^\lambda \right)^{N/m} \\ &= \left((\theta s)^{m-m^*} (\log(\theta s))^\lambda \right)^{N/m} \theta^{(m^*-m)\frac{N}{m}} \end{aligned}$$

and we conclude that

$$\inf_{s_1, s_2 \in [\theta s, s]} Q(s_2) \left(\frac{s^{m-1}}{f(s_1)} \right)^{N/m} \geq \theta^{(m^*-m)\frac{N}{m}} Q(\theta s) \left(\frac{(\theta s)^{m-1}}{f(\theta s)} \right)^{N/m}.$$

Hence, it suffices to show that

$$\lim_{s \rightarrow \infty} Q(s) \left(\frac{s^{m-1}}{f(s)} \right)^{N/m} = \infty.$$

From the definition of f , it holds that there exist a constant $C_0 > 0$ such that for $s > s_0$,

$$F(s)(\log(s))^\lambda = C_0 + \frac{s^{m^*}}{m^*} + \lambda \int_{s_0}^s \frac{F(t)}{t} (\log(t))^{\lambda-1} dt,$$

hence

$$\begin{aligned} NmF(s) - (N-m)sf(s) &= (\log(s))^{-\lambda} \left(NmF(s)(\log(s))^\lambda - (N-m)s^{m^*} \right) \\ &= (\log(s))^{-\lambda} \left(C_0 Nm + Nm\lambda \int_{s_0}^s \frac{F(t)}{t} (\log(t))^{\lambda-1} dt \right) \end{aligned}$$

and thus

$$Q(s) \left(\frac{s^{m-1}}{f(s)} \right)^{N/m} = \frac{C_0 Nm + Nm\lambda \int_{s_0}^s \frac{F(t)}{t} (\log(t))^{\lambda-1} dt}{s^{m^*} (\log(s))^{\lambda(m-N)/m}}.$$

Since the denominator in this expression tends to infinity as $s \rightarrow \infty$, we may apply L'Hospital's rule to obtain that

$$\begin{aligned}
\lim_{s \rightarrow \infty} Q(s) \left(\frac{s^{m-1}}{f(s)} \right)^{N/m} &= \lim_{s \rightarrow \infty} \frac{\lambda N m F(s)}{s^{m^*} (m^* (\log(s))^{1-\frac{\lambda N}{m}} + \lambda (\frac{m-N}{m}) (\log(s))^{-\frac{\lambda N}{m}})} \\
&= \lim_{s \rightarrow \infty} \frac{\lambda N m F(s)}{s^{m^*} (\log(s))^{-\lambda N/m} (m^* \log(s) + \lambda (\frac{m-N}{m}))} \\
&= \lim_{s \rightarrow \infty} \frac{\lambda N m f(s)}{(s^{m^*} (\log(s))^{-\frac{\lambda N}{m}})' (m^* \log(s) + \lambda (\frac{m-N}{m})) + m^* s^{m^*-1} (\log(s))^{-\frac{\lambda N}{m}}} \\
&= \lim_{s \rightarrow \infty} \frac{\lambda N m (\log(s))^{1-\lambda+\frac{\lambda N}{m}}}{(m^* \log(s) - \lambda N/m) (m^* \log(s) + \lambda (\frac{m-N}{m})) + m^* \log(s)}
\end{aligned}$$

As $1 - \lambda + \frac{\lambda N}{m} > 2$ by assumption $\lambda > m/(N-m)$, the result follows and we may apply Theorem 1.1 to obtain that problem (P) has solutions with any arbitrary number of nodes.

We end this article by giving an example of a nonlinearity for which γ^+ and $-\gamma^-$ are finite, and an example for which $\gamma^+ < \infty$ and $\gamma^- = -\infty$.

Let $g_1 : \mathbb{R} \rightarrow \mathbb{R}$ be the odd extension of

$$g_1(x) = \begin{cases} x^3 - \sqrt{2}x & \text{if } x \in [0, 2], \\ 8 - x & \text{if } x \in [2, 8], \\ h(x) & \text{if } x \in [8, \infty), \end{cases}$$

and let $g_2 : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$g_2(x) = \begin{cases} 6|x+1|^{-1/3}(x+1)^{-1} & \text{if } x \in (-\infty, -2), \\ x^3 - x & \text{if } x \in [-2, 2], \\ 8 - x & \text{if } x \in [2, 8], \\ h(x) & \text{if } x \in [8, \infty), \end{cases}$$

where h is *any continuous* function such that $h(8) = 0$. Note that g_1 satisfies assumptions (f_1) , (f_2) and (f_4) , with $\gamma^- = -8$, $\gamma^+ = 8$, $\beta^- = -\left(\frac{128}{9}\right)^{1/5}$, $\beta^+ = \left(\frac{128}{9}\right)^{1/5}$, and g_2 satisfies assumptions (f_1) , (f_2) , (f_3) and (f_4) with $\gamma^+ = 8$, $\beta^+ = \sqrt{2}$, $\beta^- = -\sqrt{2}$ and $\gamma^- = -\infty$. Note that *no restriction* on the growth of h at infinity is needed.

By Theorem 1.1, we conclude that for $i = 1, 2$, the problem

$$\Delta u + g_i(u) = 0, \quad x \in \mathbb{R}^N, \quad \lim_{|x| \rightarrow \infty} u(x) = 0,$$

has radially symmetric solutions having any prescribed number of nodes.

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